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New properties of hypergeometric series derivable from Feynman integrals: II. A generalisation of the H function

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Abstract. Further examples of the use of Feynman integrals enable the derivation of new properties of hypergeometric series including new analytic continuation formulae for a generalised hypergeometric series and for a Kampé de Fériet function. This motivates the derivation of two new summation formulae for a generalised hypergeometric series and furthermore leads to a natural generalisation of the H function. While the latter, as is well known, contains as particular cases most of the special functions of applied mathematics, it does not contain some of importance, for instance the Riemann zeta function nor indeed any polylogarithm. Our generalisation of the H function does contain the polylogarithm; it also contains the exact partition function of the Gaussian model from statistical mechanics. Another new result is the simple summation formula

$$_3F_2[1, 1, 3/2; 2, 2; x] = -4x^{-1} \ln\{[1+(1-x)^{1/2}]/2\}.$$

1. Introduction

In Inayat-Hussain (1987, hereafter referred to as I) we demonstrated the usefulness of Feynman integrals in enabling the derivation of new transformation, summation and reduction formulae for single and multiple variable hypergeometric series.

Here we extend the work by obtaining two new analytic continuation formulae for hypergeometric series. The first of these leads to two new summation formulae and also points to a new generalisation of the H function. The latter, while continually finding applications in mathematics (see, e.g., Schneider 1986 and references therein), is however of limited use in the field of statistical mechanics. Our generalisation of the H function does have an interesting application in statistical mechanics, as discussed below.

The new results given in this paper are obtained from evaluating in two ways certain Feynman integrals which arise in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions (Ma 1976, Inayat-Hussain and Buckingham 1986). For convenience, these integrals are listed in table 1, together with their convergence conditions. The notation used here is that given in § 2 of I.

In this paper we first obtain, by the use of the integral g, a new analytic continuation formula (with $z \to 1/z$) for a $_pF_{p-1}(z)$ from the domain $|z|<1$ to the domain with $|\arg z| < \pi$. A special case of this formula is a new analytic continuation formula (2) for Gauss' hypergeometric function. In § 2.2 the integral h is used to deduce, for a Kampé de Fériet function, an analytic continuation formula from the domain $|z|<1$ to the domain $|\arg(1-z)| < \pi$, with $z \to z/(z-1)$.

Section 3 contains the derivations of summation formulae for certain single variable hypergeometric functions.

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Table 1. Some integrals and their convergence conditions. These integrals, with the exception of h, are over d-dimensional space with $p_i (i = 1, \ldots, d)$ the components of the vector $\boldsymbol{p}$. The region R is the half-space in which $\boldsymbol{p} \cdot \boldsymbol{1} \geq 0$; $\boldsymbol{1}$ is an arbitrary unit vector with components $1_i (i = 1, \ldots, d)$.

$$f_{1/2} \equiv f_{1/2}(\alpha, \beta, \nu, d) = (2\pi)^{-d} \int_R \mathrm{d}\boldsymbol{p} |\boldsymbol{p}|^{-\alpha} |\boldsymbol{p} \cdot \boldsymbol{1}|^{\nu} |\boldsymbol{p} + \boldsymbol{1}|^{-\beta}$$

where $\alpha < d + \nu < \alpha + \beta$ and $\nu > -1$.

$$f \equiv f(\alpha, \beta, \nu, d) = (2\pi)^{-d} \int \mathrm{d}\boldsymbol{p} |\boldsymbol{p}|^{-\alpha} |\boldsymbol{p} \cdot \boldsymbol{1}|^{\nu} |\boldsymbol{p} + \boldsymbol{1}|^{-\beta}$$

where $\alpha < d + \nu < \alpha + \beta$, $\nu > -1$ and $\beta < d$.

$$g \equiv g(\gamma, \eta, \mu, m; z) = (2\pi)^{-d} \int_{|\boldsymbol{p}| \leq 1} \mathrm{d}\boldsymbol{p} |\boldsymbol{p}|^{2\eta - d} (\ln |\boldsymbol{p}|)^m |\boldsymbol{p} z^{1/2} + \boldsymbol{1}|^{-2\gamma}$$

where $|\arg z| < \pi$, $0 < \eta < \gamma$, $\mu > \max\{2\gamma - 2, -1\}$, $\mu = d - 2$ and $m = 0, 1, 2, 3, \ldots$.

$$\pi \equiv \pi(d, \sigma; r, \boldsymbol{1}) = (2\pi)^{-d} \int \mathrm{d}\boldsymbol{p} \left(r + \sum_{i=1}^{d} |p_i|^{\sigma} \right)^{-1} \left(r + \sum_{i=1}^{d} |p_i + 1_i|^{\sigma} \right)^{-1}$$

where $0 < \sigma < d < 2\sigma$ and $r \geq 0$.

$$h \equiv h(\mu; \beta) = \int_0^{\infty} \mathrm{d}x\, x^{2\mu - 1} (x^2 + \beta)^{-1} \tan^{-1} x$$

where $|\arg \beta| < \pi$ and $-\frac{1}{2} < \mu < 1$.

Section 4 contains the generalisation of the H function. This emerged from attempts to express in terms of the H function, both the integral g for non-integer m and the free energy of the Gaussian model (see, e.g., Joyce 1972) in arbitrary dimensions.

Finally, in § 5, we summarise and briefly discuss the results together with those of I.

2. Analytic continuation formulae

2.1. A generalised hypergeometric function

The integral g (see table 1) permits the derivation of a new analytic continuation formula (with z going to $1/z$) for a generalised hypergeometric function from the domain $|z| < 1$ to the domain $|\arg z| < \pi$. The analysis below shows that

$$\begin{aligned} {}_{3+m}F_{2+m}\left[\begin{matrix} \gamma, \eta, \ldots, \eta, \gamma - \mu/2; \\ 1 + \eta, \ldots, 1 + \eta, 1 + \mu/2; \end{matrix}\; z\right] &= \frac{(-1)^{1+m} \eta^{1+m} z^{-\gamma}}{(\gamma - \eta)^{1+m}}\, {}_{3+m}F_{2+m}\left[\begin{matrix} \gamma, \gamma - \eta, \ldots, \gamma - \eta, \gamma - \mu/2; \\ 1 + \gamma - \eta, \ldots, 1 + \gamma - \eta, 1 + \mu/2; \end{matrix}\; z^{-1}\right] \\ &+ \frac{(-1)^m \eta^{1+m} \Gamma(1 + \mu/2) \Gamma(1 + \mu/2 - \gamma)}{\Gamma(1 + m) \Gamma(\gamma)} \\ &\times \frac{\partial^m}{\partial \eta^m} \left(\frac{z^{-\eta} \Gamma(\eta) \Gamma(\gamma - \eta)}{\Gamma(1 + \eta - \gamma + \mu/2) \Gamma(1 - \eta + \mu/2)} \right) \end{aligned} \tag{1}$$

where $0 < \eta < \gamma$ and $\mu > \max\{2\gamma - 2, -1\}$.

A special case of (1), obtained by setting $\mu = \gamma - 1$ and $m = 0$, is a new linear transformation formula for a ${}_2F_1$ (for an extensive list of the linear transformation formulae for this function, see Abramowitz and Stegun (1972, ch 15)):

$$
{}_2F_1[\gamma, \eta; 1+\eta, z] = -\eta(\gamma-\eta)^{-1} z^{-\gamma} {}_2F_1[\gamma, \gamma-\eta; 1+\gamma-\eta; z^{-1}]
$$

$$
+\frac{\eta\Gamma(\gamma/2+1/2)\Gamma(1/2-\gamma/2)\Gamma(\eta)\Gamma(\gamma-\eta)z^{-\eta}}{\Gamma(\gamma)\Gamma(1/2-\gamma/2+\eta)\Gamma(1/2+\gamma/2-\eta)} \tag{2}
$$

where $0<\eta<\gamma<1$, $|\arg z|<\pi$.

For $z=1$, (2) reduces to an identity between two ${}_2F_1(1)$. This identity is not new; it can be shown to be a consequence of Gauss' summation formula (Slater 1966, equation (III.3)). Further these ${}_2F_1(1)$ can be combined to yield a ${}_4F_3(1)$, but the resulting summation formula is again not new; this time it is a special case of a well known summation formula for a ${}_5F_4(1)$ (Slater 1966, equation (III.12)) in which a cancellation takes place between one numerator and one denominator parameter resulting in the ${}_4F_3(1)$. The special case of the more general identity (1) obtained by setting $z=1$ does, however, turn out to have a non-trivial consequence in that it motivates the derivation of new summation formulae for the generalised hypergeometric function with arguments 1 or -1, an aspect to be developed in § 3.

The proof of (1) is based on evaluating the integral g in two different ways:

$$
g = \frac{K_{d-1}}{2\pi}\int_0^1 \mathrm{d}p \int_0^\pi \mathrm{d}\theta\, G(p,\theta) \tag{3}
$$

$$
= \frac{K_{d-1}}{2\pi}\int_0^\pi \mathrm{d}\theta \int_0^\infty \mathrm{d}p\, G(p,\theta) - \frac{K_{d-1}}{2\pi}\int_1^\infty \mathrm{d}p \int_0^\pi \mathrm{d}\theta\, G(p,\theta) \tag{4}
$$

where

$$
G(p,\theta) \equiv p^{2\eta-1}(\ln p)^m(\sin\theta)^\mu(1+2z^{1/2}p\cos\theta+zp^2)^{-\gamma}
$$

and the factor

$$
K_d \equiv 2^{1-d}\pi^{-d/2}/\Gamma(d/2) \tag{5}
$$

is the ratio of the surface area of a unit sphere in d dimensions to $(2\pi)^d$. In (4) g has been written as the integral over all space less the integral outside the unit sphere and the order in which the integrals over p and θ have been written indicates the way in which these integrals can be readily evaluated.

Standard integrals (Gradshteyn and Ryzhik 1980, § 3.665, equation (2) and § 4.272, equation (6)) can be used in (3) to yield the left-hand side of (1), to within a few factors of the gamma function.

To evaluate the integral over all space on the right of (4), we first write

$$
p^{2\eta-1}(\ln p)^m \qquad \text{as} \qquad (\partial/\partial(2\eta))^m p^{2\eta-1}
$$

and by a suitable change of variable extract from the integral the dependence on z (which is of the form $z^{-\eta}$). The resulting double integral can now be evaluated (Gradshteyn and Ryzhik 1980, § 3.252, equation (10) and § 7.166) in the given order to yield the derivative term on the right of (1). The evaluation of the other double integral on the right of (4) is similar to that of the double integral in (3); this yields the hypergeometric function on the right of (1), thus completing the proof.

2.2. A Kampé de Fériet function

There are various analytic continuation formulae known for the ${}_2F_1$ hypergeometric function and the Appell functions (Slater 1966, ch 8). However, little is known about such formulae for the Kampé de Fériet function, with the exception of those special cases which reduce to lower 'order' hypergeometric functions (see, e.g., Exton 1976, § 1.5, § 4.7.1, 1978, § 1.3.2, Buschman and Srivastava 1982, Karlsson 1984). (The term 'order', as used here, refers to a classification in terms of the number of variables or the number (and types) of Pochhammer symbols involving the parameters. The classification by order has been shown to break down (see Carlson 1976) and remains an incompletely resolved problem in the theory of hypergeometric functions (see Srivastava and Karlsson 1985, ch 9).)

A new result on the analytic continuation of the Kampé de Fériet function is the following:

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix}\mu, 1: \mu; 1; \\ 1+\mu, 2: -; -;\end{matrix}\; z, 1\right] = \frac{\mu^2}{(1-z)(1-\mu)^2} F^{2:1;1}_{2:0;0}\left[\begin{matrix}1-\mu, 1: 1-\mu; 1; \\ 2-\mu, 2: -; -;\end{matrix}\; z/(z-1), 1\right]$$
$$+\frac{\mu}{(1-z)^{\mu}(1-\mu)}[\pi\cot(\pi\mu)+\ln(1-z)] \qquad (6)$$

where $|\arg(1-z)|<\pi$ and $-1/2<\mu<1$. The proof of this formula is based on evaluating in two ways a remarkably simple integral, namely h in table 1. After writing $\tan^{-1} x$, which appears in the integrand, as an inverse Mellin integral (Marichev 1983), an interchange of the order of integration of the resulting double integral followed by the evaluation of the x integral yields

$$h(\mu;\beta) = \frac{\pi^2}{4\beta^{1-\mu}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \frac{\beta^{-s/2}}{s\cos(\pi s/2)\sin[\pi(s/2-\mu)]} \qquad (7)$$

where $-1<c<0$. For $|\beta|>1$, (7) can be evaluated by summing the residues at the simple poles ($s=0$, $s=1+2n$, $s=2\mu+2n$, $n=0,1,2,\ldots$) in the right half of the complex s plane, whereas for $|\beta|<1$ the contour of integration is closed around the poles ($s=-1-2n$, $s=2(\mu-1-n)$, $n=0,1,2,\ldots$) in the left half of the complex s plane. The contribution from the semi-circular part of the closed contour which arises in both cases can be shown to vanish in the limit as its radius approaches infinity. These steps give, for $|\beta|>1$,

$$h(\mu;\beta) = \frac{\pi^2}{4\beta^{1-\mu}\sin(\pi\mu)} + \frac{\pi}{2\beta^{3/2-\mu}\cos(\pi\mu)}$$
$$\times\left({}_2F_1[1, 1/2; 3/2; 1/\beta] - \frac{\beta^{1/2-\mu}}{2\mu}\,{}_2F_1[1, \mu; 1+\mu; 1/\beta]\right) \qquad (8)$$

and for $|\beta|<1$,

$$h(\mu;\beta) = \frac{\pi}{2\beta^{1/2-\mu}\cos(\pi\mu)}\left({}_2F_1[1, 1/2; 3/2; \beta] - \frac{\beta^{1/2-\mu}}{2(1-\mu)}\,{}_2F_1[1, 1-\mu; 2-\mu; \beta]\right). \qquad (9)$$

The hypergeometric functions in (8) and (9) are of the form ${}_2F_1[a, b; a+b; z]$ and we can exploit some known results (Abramowitz and Stegun 1972, equations (15.1.4) and (15.3.10)) to extend their domains of analyticity. Thus we obtain, with $\psi(z)$ the digamma function,

$$h(\mu;\beta)=\frac{\pi^2}{4\beta^{1-\mu}\sin(\pi\mu)}+\frac{\pi\ln(1+\beta^{-1/2})}{2\beta^{1-\mu}\cos(\pi\mu)}$$
$$-\frac{\pi}{4\beta\cos(\pi\mu)}\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}[\psi(1+n)-\psi(\mu+n)](1-\beta^{-1})^n \qquad (10)$$

valid for $|\arg(1-\beta^{-1})|<\pi$ and $|1-\beta^{-1}|<1$ and

$$h(\mu;\beta)=\frac{\pi\ln(1+\beta^{1/2})}{2\beta^{1-\mu}\cos(\pi\mu)}-\frac{\pi}{4\cos(\pi\mu)}\sum_{n=0}^{\infty}\frac{(1-\mu)_n}{n!}[\psi(1+n)-\psi(1-\mu+n)](1-\beta)^n \qquad (11)$$

valid for $|\arg(1-\beta)|<\pi$ and $|1-\beta|<1$. The domains of analyticity of (10) and (11) clearly overlap. Inserting the series representation for the digamma functions (Gradshteyn and Ryzhik 1980, § 8.363, equation (3)) and making use of the definition of the Kampé de Fériet function (see I, equation (3)), equating (10) and (11) finally leads to the result (6), where z is to be identified with $1-\beta^{-1}$.

A similar calculation for the integral

$$h_1(\mu;\beta)=\int_0^\infty \mathrm{d}x\, x^{\mu-1}(x+\beta)^{-2}\ln(1+x)$$

with $-1<\mu<2$, yields the further result

$$F^{2:\,1;\,1}_{2:\,0;\,0}\left[{\mu,\,1:\,\mu;\,1; \atop 1+\mu,\,2:\,-;\,-;}\ z,1\right]=\frac{-\mu}{(2-\mu)(1-z)^2}F^{2:\,1;\,1}_{2:\,0;\,0}\left[{2-\mu,\,1:\,2-\mu;\,1 \atop 3-\mu,\,2:\,-;\,-;}\ z/(z-1),1\right]$$
$$+\frac{\mu}{1-\mu}\left(\frac{1}{(1-\mu)(1-z)}+\frac{\pi\cot(\pi\mu)+\ln(1-z)}{(1-z)^\mu}\right) \qquad (12)$$

where $|\arg(1-z)|<\pi$ and, as before, $z\equiv 1-\beta^{-1}$. This formula is related to (6); in fact it can be derived directly by applying the 'contiguous relation' (Abramowitz and Stegun 1972, ch 15)

$$\beta\, {}_2F_1[1,2-\mu;3-\mu;\beta]={}_2F_1[1,1-\mu;2-\mu;\beta]-1$$

to (9) and repeating the steps which led to (6).

3. Summation formulae for generalised hypergeometric series

Consider the following summation formula obtainable from (1) by setting $z=1$ and, for the moment, $m=0$:

$${}_5F_4\left[{\gamma,\,1+\gamma/2,\,\gamma-\mu/2,\,\eta,\,\gamma-\eta; \atop \gamma/2,\,1+\mu/2,\,1+\eta,\,1+\gamma-\eta;}\ 1\right]=\frac{\Gamma(1+\mu/2)\Gamma(1-\gamma+\mu/2)\Gamma(1+\eta)\Gamma(1+\gamma-\eta)}{\Gamma(1+\gamma)\Gamma(1+\eta-\gamma+\mu/2)\Gamma(1+\mu/2-\eta)}. \qquad (13)$$

This formula is not new but is a special case of a well known summation formula for a well poised ${}_5F_4(1)$ (see Slater 1966, equation (III.12)). A generalised hypergeometric function ${}_{1+p}F_p$ is said to be well poised (Whipple 1926) when its parameters $a_0, a_1, \ldots, a_p$ and $b_0, b_1, \ldots, b_p$ or appropriate permutations are such that $1+a_0=a_1+b_1=\ldots=a_p+b_p$.

However, a new result does arise in the special case obtained from (1) by setting $m=1$ (with $z=1$). This yields, for a well poised ${}_7F_6(1)$ with certain denominator parameters greater by unity than the numerator parameters, the formula

$$
{}_7F_6\left[{\gamma, 1+\gamma/2, \gamma-\mu/2, \eta, \eta, \gamma-\eta, \gamma-\eta; \atop \gamma/2, 1+\mu/2, 1+\eta, 1+\eta, 1+\gamma-\eta, 1+\gamma-\eta;} 1\right]
$$

$$
= -\frac{\eta^2(\gamma-\eta)^2\Gamma(\eta)\Gamma(\gamma-\eta)\Gamma(1+\mu/2)\Gamma(1+\mu/2-\gamma)}{(\gamma-2\eta)\Gamma(1+\gamma)\Gamma(1+\mu/2+\eta-\gamma)\Gamma(1+\mu/2-\eta)}
$$

$$
\times[\psi(\eta)+\psi(1+\mu/2-\eta)-\psi(\gamma-\eta)-\psi(1+\mu/2+\eta-\gamma)]. \quad (14)
$$

Karlsson (1971, 1974) has shown that generalised hypergeometric series with integral differences between certain numerator and denominator parameters (and provided the parameters are distinct) can be written in terms of finite sums over lower-order functions (generalisations to non-integral differences can be found in Chakrabarty (1974) and Panda (1976)). Unfortunately the usefulness of his results (and also those of Chakrabarty and Panda) is limited by the fact that they do not preserve the symmetry properties (for example, the property of being well poised) of the hypergeometric functions. To give an example, application of equation (5) of Karlsson (1974) to the ${}_5F_4(1)$ in (13) would yield the following reduction:

$$
{}_5F_4\left[{\gamma, 1+\gamma/2, \gamma-\mu/2, \eta, \gamma-\eta; \atop \gamma/2, 1+\mu/2, 1+\eta, 1+\gamma-\eta;} 1\right]
$$

$$
= \frac{1}{\gamma-2\eta}\left((\gamma-\eta)\,{}_4F_3\left[{\gamma, 1+\gamma/2, \gamma-\mu/2, \eta; \atop \gamma/2, 1+\mu/2, 1+\eta;} 1\right] - \eta\,{}_4F_3\left[{\gamma, 1+\gamma/2, \gamma-\mu/2, \gamma-\eta; \atop \gamma/2, 1+\mu/2, 1+\gamma-\eta;} 1\right]\right). \quad (15)
$$

The two ${}_4F_3(1)$ in (15) cannot be summed by any of the known summation formulae whereas the ${}_5F_4(1)$, as demonstrated by (13), is in fact summable. Thus, in this example, the use of equation (5) of Karlsson (1974) would only obscure the simplicity of the problem. Furthermore Karlsson's formula cannot be applied directly (that is, without getting involved with derivatives with respect to parameters) to the ${}_7F_6(1)$ in (14), as some of its parameters, namely those having counterparts differing by integer values, have the same value.

A direct proof of (14) would take advantage of the well poised structure of the ${}_7F_6(1)$. This has been achieved and further generalised in the form of the following new summation formulae for certain well poised hypergeometric functions of argument 1 or -1:

$$
{}_{3+2m}F_{2+2m}\left[{\gamma, 1+\gamma/2, 1-a, \eta_1, \gamma-\eta_1, \ldots, \eta_m, \gamma-\eta_m; \atop \gamma/2, \gamma+a, 1+\eta_1, 1+\gamma-\eta_1, \ldots, 1+\eta_m, 1+\gamma-\eta_m;} 1\right]
$$

$$
= \frac{(-1)^{1+m}\Gamma(a)\Gamma(\gamma+a)\prod_{j=1}^{m}[\eta_j(\gamma-\eta_j)]}{\Gamma(1+\gamma)}
$$

$$
\times\sum_{j=1}^{m}\frac{\Gamma(\gamma-\eta_j)\Gamma(\eta_j)}{\Gamma(a+\eta_j)\Gamma(\gamma+a-\eta_j)\prod_{k=1,k\neq j}^{m}(\gamma-\eta_j-\eta_k)(\eta_j-\eta_k)} \quad (16)
$$

provided $a>1-m$, $a\neq 0, -1, -2, -3, \ldots$, and $\gamma-\eta_j$ and η_j are not negative integers for any j;

$$
{}_{2+2m}F_{1+2m}\left[{\gamma, 1+\gamma/2, \eta_1, \gamma-\eta_1, \ldots, \eta_m, \gamma-\eta_m; \atop \gamma/2, 1+\eta_1, 1+\gamma-\eta_1, \ldots, 1+\eta_m, 1+\gamma-\eta_m;} -1\right]
$$

$$
= \frac{(-1)^{1+m}\prod_{i=1}^{m}[\eta_i(\gamma-\eta_i)]}{\Gamma(1+\gamma)}\sum_{j=1}^{m}\frac{\Gamma(\gamma-\eta_j)\Gamma(\eta_j)}{\prod_{k=1,k\neq j}^{m}(\gamma-\eta_j-\eta_k)(\eta_j-\eta_k)} \quad (17)
$$

provided $\gamma<2m$ and $\gamma-\eta_j$ and η_j are not negative integers for any j.

The proofs of (16) and (17) are based on induction on m and while we give in some detail the proof of (16) we only indicate that of (17).

Let $\Phi(m)$ denote the left-hand side of (16). For $m=0$, (16) is valid by Dixon's theorem (Slater 1966). Suppose now that (16) is valid for $m-1$. By introducing the series representation (1) of I and making use of the partial fraction expansion

$$\frac{1}{(\eta_{m-1}+n)(\gamma-\eta_{m-1}+n)(\eta_m+n)(\gamma-\eta_m+n)} = \frac{1}{(\eta_m-\eta_{m-1})(\gamma-\eta_m-\eta_{m-1})} \times\left(\frac{1}{(\eta_{m-1}+n)(\gamma-\eta_{m-1}+n)}-\frac{1}{(\eta_m+n)(\gamma-\eta_m+n)}\right) \tag{18}$$

$\Phi(m)$ can be written as a difference of two $\Phi(m-1)$ which by the hypothesis can be summed. Thus

$$\Phi(m)=\frac{(-1)^m\Gamma(a)\Gamma(\gamma+a)\prod_{i=1}^m[\eta_i(\gamma-\eta_i)]}{\Gamma(1+\gamma)(\eta_m-\eta_{m-1})(\gamma-\eta_m-\eta_{m-1})} \times\left(\sum_{j=1}^{m-1}\frac{\Gamma(\gamma-\eta_j)\Gamma(\eta_j)}{\Gamma(a+\eta_j)\Gamma(\gamma+a-\eta_j)\prod_{k=1,k\neq j}^{m-1}(\gamma-\eta_j-\eta_k)(\eta_j-\eta_k)} - \sum_{j=1,j\neq m-1}^{m}\frac{\Gamma(\gamma-\eta_j)\Gamma(\eta_j)}{\Gamma(a+\eta_j)\Gamma(\gamma+a-\eta_j)\prod_{k=1,k\neq j}^{m}(\gamma-\eta_j-\eta_k)(\eta_j-\eta_k)}\right). \tag{19}$$

Each of the sums appearing on the right-hand side of (19) can be split to yield a sum running over j from 1 to $m-2$ together with a single term. The resulting expressions are then combined in a way which effectively reverses the partial fraction expansion (18) to yield the right-hand side of (16). Appeal to the principle of induction completes the proof.

The proof of (17) is also based on the same partial fraction expansion (18). The formula for $m=0$ is true by Kummer's theorem (Slater 1966) and the result for general m is deduced in the same manner as above.

4. A generalisation of the H function

In this section we discuss certain functions which are not themselves members of the class of functions included in the H function but which naturally suggest a certain generalisation (26) of that function.

Consider the function

$$g_1\equiv(-1)^m g(\gamma,\eta,\mu,m;z)=(2\pi)^{-d}\int_{|\boldsymbol{p}|\leq 1}\mathrm{d}\boldsymbol{p}|\boldsymbol{p}|^{2\eta-d}[\ln(1/|\boldsymbol{p}|)]^m|\mathbf{1}+z^{1/2}\boldsymbol{p}|^{-2\gamma}$$

where g is the integral dealt with in § 2.1. This function is real valued, even for non-integer values of m, if z is real.

By using the same method as in § 2, it is easy to show that

$$g_1=\frac{K_{d-1}m!B(1/2,1/2+\mu/2)}{2^{2+m}\pi}\sum_{n=0}^{\infty}\frac{(\gamma)_n(\gamma-\mu/2)_n z^n}{(1+\mu/2)_n(\eta+n)^{1+m}n!} \tag{20}$$

where K_d has been defined in (5). The series in (20), which for integer values of m represents a generalised hypergeometric function, can be written as a Mellin integral so that

$$g_1 = \frac{K_{d-1} m! \Gamma(1+\mu/2) B(1/2, 1/2+\mu/2)}{2^{2+m}\pi\Gamma(\gamma)\Gamma(\gamma-\mu/2)} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{(-z)^s\Gamma(-s)\Gamma(\gamma+s)\Gamma(\gamma-\mu/2+s)}{(\eta+s)^{1+m}\Gamma(1+\mu/2+s)}.$$

For non-integer values of m, this Mellin integral is clearly not a H function. The latter is defined by

$$H^{m,n}_{p,q}\left[z\left|{(\alpha_1, A_1), \ldots, (\alpha_p, A_p) \atop (\beta_1, B_1), \ldots, (\beta_q, B_q)}\right.\right] = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{z^s \prod_{j=1}^m \Gamma(\beta_j - B_j s) \prod_{j=1}^n \Gamma(1-\alpha_j + A_j s)}{\prod_{j=1+m}^q \Gamma(1-\beta_j + B_j s) \prod_{j=1+n}^p \Gamma(\alpha_j - A_j s)}. \tag{21}$$

The conditions for absolute convergence of the integral in (21) are discussed in Srivastava *et al* (1982).

In equilibrium statistical mechanics a simple model of phase transitions is the Gaussian model (Joyce 1972) which furnishes yet another example of a function which is not a H function. The free energy of this model on a Bravais lattice in d dimensions is given by

$$-\beta F(d; \beta, \xi) = \tfrac{1}{2} \ln \xi - \tfrac{1}{2}(2\pi)^{-d} \int_0^{2\pi} \ldots \int_0^{2\pi} d\boldsymbol{k} \ln(\xi - \tfrac{1}{2}\beta \hat{J}(\boldsymbol{k}))$$

where $\hat{J}(\boldsymbol{k})$ is the Fourier transform of the interaction energy, β is the inverse temperature and $\xi(\xi > 0)$ is an 'intensive' thermodynamic variable. For a body-centred cubic lattice with only nearest-neighbour interactions (Smith 1983, appendix 2)

$$\hat{J}(\boldsymbol{k}) = J \prod_{i=1}^{d} \cos k_i \qquad (J>0)$$

and the free energy reduces to

$$\beta F(d; \varepsilon) = \tfrac{1}{2}(2\pi)^{-d} \int_0^{2\pi} \ldots \int_0^{2\pi} d\boldsymbol{k} \ln\left(1-(1+\varepsilon)^{-1} \prod_{i=1}^{d} \cos k_i\right). \tag{22}$$

The variable $\varepsilon = \beta_c/\beta - 1$ is a reduced temperature interval, where $\beta_c = 2\xi/J$ is the critical temperature.

By expanding out the logarithm in (22), the free energy can be expressed in terms of a series:

$$\beta F(d; \varepsilon) = -2^{-2-d}(1+\varepsilon)^{-2} \sum_{n=0}^{\infty} \frac{(1)_n[(3/2)_n]^d}{[(2)_n]^{1+d}(1+\varepsilon)^{2n}}. \tag{23}$$

For integer values of d, this gives a generalised hypergeometric function:

$$\beta F(d; \varepsilon) = -2^{-2-d}(1+\varepsilon)^{-2}{}_{2+d}F_{1+d}[1, 1, 3/2, \ldots, 3/2; 2, \ldots, 2; (1+\varepsilon)^{-2}]. \tag{24}$$

There is interest in (23) even for non-integer values of d, when the free energy is no longer a generalised hypergeometric function and is not even expressible as an H function; this is evident from the Mellin integral representation:

$$\beta F(d; \varepsilon) = -\frac{1}{4\pi^{d/2}(1+\varepsilon)^2} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \frac{[-(1+\varepsilon)^{-2}]^s \Gamma(-s)[\Gamma(1+s)]^2[\Gamma(3/2+s)]^d}{[\Gamma(2+s)]^{1+d}}.$$

In one dimension, the integral (22) can be evaluated directly by means of the residue theorem to yield

$$\beta F(1; \varepsilon) = -\tfrac{1}{2} \ln 2 + \tfrac{1}{2} \ln\{1 + [1-(1+\varepsilon)^{-2}]^{1/2}\}. \tag{25}$$

Equating (25) and (24) (with $d=1$), and defining $x=(1+\varepsilon)^{-2}$, gives the new summation formula for a ${}_3F_2$, as mentioned in the abstract:

$$ {}_3F_2[1,1,3/2;2,2;x]=4x^{-1}\{\ln 2-\ln[1+(1-x)^{1/2}]\}. $$

This formula has also been derived independently by Gottschalk (1986) in the course of studying solutions of the Schrödinger equation describing multielectron atoms.

A further example of a function which is not a special case of the H function is the polylogarithm of complex order ν (see, e.g., Marichev 1983).

The three examples discussed above all suggest the following generalisation of the H function which we denote by the symbol $\bar{H}$:

$$ \bar{H}^{m,n}_{p,q}\left[z\left|{(\alpha_1,A_1,a_1),\ldots,(\alpha_n,A_n,a_n),(\alpha_{1+n},A_{1+n}),\ldots,(\alpha_p,A_p) \atop (\beta_1,B_1),\ldots,(\beta_m,B_m),(\beta_{1+m},B_{1+m},b_{1+m}),\ldots,(\beta_q,B_q,b_q)}\right.\right] $$
$$ =\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\frac{z^s\prod_{j=1}^{m}\Gamma(\beta_j-B_js)\prod_{j=1}^{n}[\Gamma(1-\alpha_j+A_js)]^{a_j}}{\prod_{j=1+m}^{q}[\Gamma(1-\beta_j+B_js)]^{b_j}\prod_{j=1+n}^{p}\Gamma(\alpha_j-A_js)}. \tag{26} $$

Comparison with (21) shows by inspection that only when a_j and b_j all take integer values does $\bar{H}$ reduce to the H function. A particularly important case of the $\bar{H}$ function corresponds to that of the generalised hypergeometric function in the case of the H function, namely

$$ \bar{H}^{1,p}_{p,1+q}\left[-z\left|{(1-\alpha_1,1,a_1),\ldots,(1-\alpha_p,1,a_p) \atop (0,1),(1-\beta_1,1,b_1),\ldots,(1-\beta_q,1,b_q)}\right.\right] $$
$$ =\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\frac{(-z)^s\Gamma(-s)\prod_{j=1}^{p}[\Gamma(\alpha_j+s)]^{a_j}}{\prod_{j=1}^{q}[\Gamma(\beta_j+s)]^{b_j}} $$
$$ =\frac{\prod_{j=1}^{p}[\Gamma(\alpha_j)]^{a_j}}{\prod_{j=1}^{q}[\Gamma(\beta_j)]^{b_j}}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}[(\alpha_j)_n]^{a_j}z^n}{\prod_{j=1}^{q}[(\beta_j)_n]^{b_j}n!}. \tag{27} $$

The three functions described above in this section, which in general are not special cases of the H function, are nevertheless covered by (27) and are therefore cases of the $\bar{H}$ function (26). In particular the Gaussian model free energy is given by

$$ \beta F(d;\varepsilon)=-\frac{(1+\varepsilon)^{-2}}{4\pi^{d/2}}\bar{H}^{1,3}_{3,2}\left[-(1+\varepsilon)^{-2}\left|{(0,1,1),(0,1,1),(-1/2,1,d) \atop (0,1),(-1,1,1+d)}\right.\right]. \tag{28} $$

5. Conclusion

Certain Feynman integrals have been shown to be a rich source of new results for hypergeometric functions of one or more variables. We have also illustrated by the use of examples, in particular from statistical mechanics, the inadequacy of the class of functions described by the H function and put forward a natural generalisation to the $\bar{H}$ function (26) but have not yet fully explored its analytic properties.

The d-dimensional integrals dealt with in both parts of the present work all have integrands which are rotationally invariant at least about one axis, thus enabling the use of the spherical coordinate transformation. The reader will by now have realised that we have not made use of the integral π (see table 1), which has only a discrete symmetry. This very property, although undesirable from the viewpoint of ease of analysis, enables the integral to be a source of new results for the Wright function and the generalised Lauricella series of n variables. In the special case of two variables, we have obtained the new summation formula

$$\sum_{n=0}^{\infty} \frac{(1)_n(9/4-d/4)_n(8-d)_{3n}2^{3n}}{(2)_n(9-d)_{4n}}\, {}_3F_2\left[\begin{matrix}-n,\ 1/2+n/2,\ 1+n/2; \\ d/2-7/2-3n/2,\ d/2-3-3n/2;\end{matrix}\ -1\right]$$

$$= \frac{\pi(8-d)\sin[\pi(d/4-1/4)]}{2^{3/2}(5-d)\sin[\pi(d/4-1)]}\left(1+\frac{\sin[\pi(d/4-1)]-\frac{1}{2}4^{d/4-1}}{\cos[\pi(d/4-1)]}\right)$$

valid for $4<d<8$. The proof of this result, the details of the more general cases and their relevance to the description of an anisotropic Lifshitz critical point will be presented elsewhere.

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